# Unsteady Heating of Rayleigh-Benard Convection

B. S. Bhadauria

Department of Mathematics and Statistics, Jai Narain Vyas University, Jodhpur, India-342001
*Current address:* Department of Mathematics, Eritrean Institute of Technology,
Asmara, Eritrea (N. E. Africa)

Reprint requests to Prof. B. S. B.; E-mail: bsbhadauria@rediffmail.com

Z. Naturforsch. **59a,** 266 – 274 (2004); received April 25, 2003

The linear thermal instability of a horizontal fluid layer with time-periodic temperature distribution is studied with the help of the Floquet theory. The time-dependent part of the temperature has been expressed in Fourier series. Disturbances are assumed to be infinitesimal. Only even solutions are considered. Numerical results for the critical Rayleigh number are obtained at various Prandtl numbers and for various values of the frequency. It is found that the disturbances are either synchronous with the primary temperature field or have half its frequency. — *2000 Mathematics Subject Classification:* 76E06, 76R10.

*Key words:* Modulation; Stability; Rayleigh Number; Even Solution; Thermal Convection.

## 1. Introduction

The present paper deals with the stability of a fluid layer confined between two horizontal rigid planes, heated from below and above periodically with time. Considerable attention has been given to this problem during the last thirty years. Chandrasekhar [1] has given a comprehensive review of the stability problem with steady temperature gradient. In numerous stability problems, oscillation of the basic state has been found sometimes to have a stabilizing and sometimes a destabilizing effect.

Venezian [2] has investigated the thermal analogue of Donnelly's experiment [3] for free-free surfaces, using sinusoidal temperature profile. His theory does not yield any such finite frequency, as obtained by Donnelly, but finds that for the case of modulation only at the lower surface, the modulation would be stabilizing with maximum stabilization occurring as the frequency goes to zero. However in his explanation, it was suggested by Venezian that linear stability theory ceases to be applicable when the frequency of modulation is sufficiently small. Rosenblat and Herbert [4] have investigated the linear stability problem for free-free surfaces, using low-frequency modulation and found an asymptotic solution. Periodicity and amplitude criteria were employed to calculate the critical Rayleigh number. Rosenblat and Tanaka [5] have studied the linear stability problem for more realistic boundary conditions i.e. rigid walls, using Galerkin's procedure. A similar problem has been considered earlier by Gershuni and Zhukhovitskii [6] for a temperature profile, obeying rectangular law. Yih and Li [7] have investigated the formation of convective cells in a fluid between two horizontal rigid boundaries with time-periodic temperature distribution, using Floquet theory. They found that the disturbances (or convection cells) oscillate either synchronously or with half frequency.

Gresho and Sani [8] have studied the effect of time-variable gravity on thermal convection in a horizontal fluid layer with rigid boundaries. They found that gravitational modulation could significantly affect the stability limits of the system. Finucane and Kelly [9] have carried out an analytical-experimental investigation to confirm the results of Rosenblat and Herbert. Besides investigating the linear stability, Roppo et al. [10] have also carried out the weakly non-linear analysis of the problem. Aniss et al. [11] have worked out a linear problem of the convection parametric instability in the case of a Newtonian fluid confined in a Hele-Shaw cell and subjected to the vertical periodic motion. In their asymptotic analysis they have investigated the influence of the gravitational modulation on the instability threshold. Recently Bhadauria [12], Bhadauria and Bhatia [13], and Bhatia and Bhadauria [14, 15] have investigated the convective instability using Saw-tooth function, Step-function and Day-night function for the modulation of the wall temperatures.

The object of the present study is to find the critical conditions under which the onset of convection takes place. To modulate the wall temperatures, we consider a temperature profile which is similar to the variation of the atmospheric temperature near to the earth's surface during one complete day-night cycle. The temperature profile has been expressed in Fourier series, which includes both sine and cosine terms. Thus keeping in mind its practical importance and to generalize the study, the above profile is considered. Further, this study can be used to calculate the results for any type of periodic temperature profile, which could be realized just by replacing the values of the Fourier coefficients. Only even solutions have been considered. The results have their relevance with convective flows in the terrestrial atmosphere.

## 2. Formulation

Consider a fluid layer of a viscous, incompressible fluid, confined between two parallel horizontal walls, one at $z=-d/2$ and the other at $z=d/2$. The walls are infinitely extended and rigid. The configuration is shown in the Figure 1.

The governing equations in the Boussinesq approximation are

$$\frac{\partial \mathbf{V}}{\partial t}+\mathbf{V}.\nabla\mathbf{V}=-\frac{1}{\rho_\mathrm{m}}\nabla p+[1-\alpha(T-T_\mathrm{m})]X+\nu\nabla^2\mathbf{V}, \tag{2.1}$$

$$\nabla.\mathbf{V}=0, \tag{2.2}$$

$$\frac{\partial T}{\partial t}+\mathbf{V}.\nabla T=\kappa\nabla^2 T, \tag{2.3}$$

where $\rho_\mathrm{m}$ and $T_\mathrm{m}$ are the reference density and temperature, respectively, $X=(0,0,-g)$, where $g$ is the acceleration due to gravity, $\nu$ is the kinematic viscosity, $\kappa$ the thermal diffusivity and $\alpha$ the coefficient of volume expansion. $\mathbf{V}=(u,v,w)$, $p$ and $T$ are, respectively, the fluid velocity, pressure and temperature, while $t$ is the time. The relation between $\rho_\mathrm{m}$ and $T_\mathrm{m}$ is given by

$$\rho=\rho_\mathrm{m}[1-\alpha(T-T_\mathrm{m})]. \tag{2.4}$$

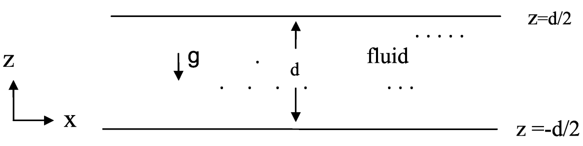


Fig. 1. Benard Configuration.

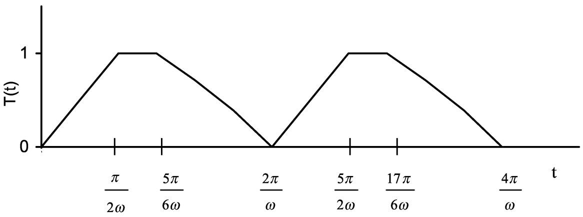


Fig. 2. Variation of the temperature $T$ with time $t$.

To modulate the wall temperatures, we consider a temperature profile as shown in Fig. 2 and defined as

$$T_\mathrm{b}(t)=\begin{cases}\dfrac{2\omega t}{\pi}, & 0\le t\le\dfrac{\pi}{2\omega},\\ 1, & \dfrac{\pi}{2\omega}\le t\le\dfrac{5\pi}{6\omega},\\ \dfrac{12}{7}\left(1-\dfrac{\omega t}{2\pi}\right), & \dfrac{5\pi}{6\omega}\le t\le\dfrac{2\pi}{\omega},\end{cases} \tag{2.5}$$

where $\omega$ is the modulating frequency and $2\pi/\omega$ is the period of oscillation. This temperature profile may be similar to the variation of the atmospheric temperature near the earth's surface during one complete day-night cycle. This is a temperature profile constructed with the idea that initially the temperature of the earth's surface is minimum (in the morning), and then it increases and attains the maximum. It remains constant for some time (in the afternoon) and then decreases slowly to reach the minimum. This completes the cycle. The profile is constructed to study a simple convection problem and not to model the atmospheric conditions. The Fourier series of the function (2.5) is given by

$$T_\mathrm{b}(t)=\frac{a_0}{2}+\sum_{k=1}^{\infty}a_k\cos k\omega t+\sum_{k=1}^{\infty}b_k\sin k\omega t, \tag{2.6}$$

where

$$a_0=\frac{14}{12}, \tag{2.6a}$$

$$a_k=\frac{2}{k^2\pi^2}\left[\frac{-10}{7}+\cos\frac{k\pi}{2}+\frac{3}{7}\cos\frac{5k\pi}{6}\right], \tag{2.6b}$$

$$b_k=\frac{2}{k^2\pi^2}\left[\sin\frac{k\pi}{2}+\frac{3}{7}\sin\frac{5k\pi}{6}\right]. \tag{2.6c}$$

The profile given above includes both sine and cosine terms, therefore it is the most general form of a periodic temperature profile. By shifting the origin we write

$$T_\mathrm{b}(t)=\sum_{k=1}^{\infty}a_k\cos k\omega t+\sum_{k=1}^{\infty}b_k\sin k\omega t, \tag{2.7}$$

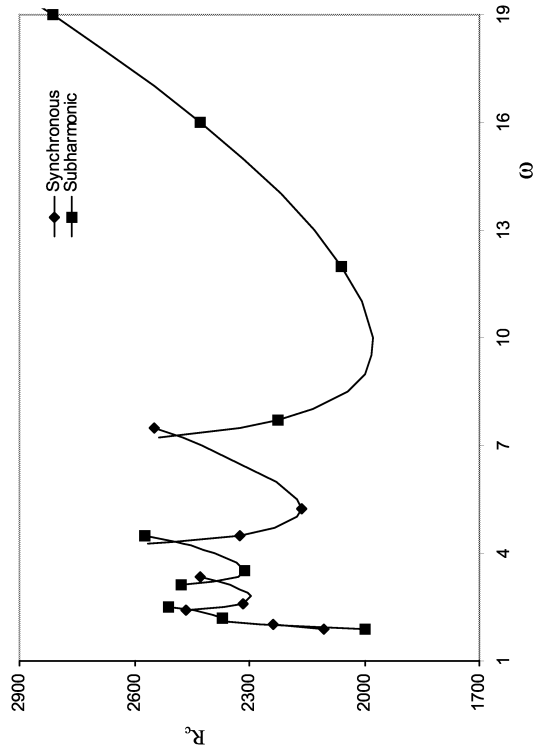


Fig. 3. Variation of $R_c$ with $\omega$. $\varepsilon = 0.1$, $P = 0.1$.

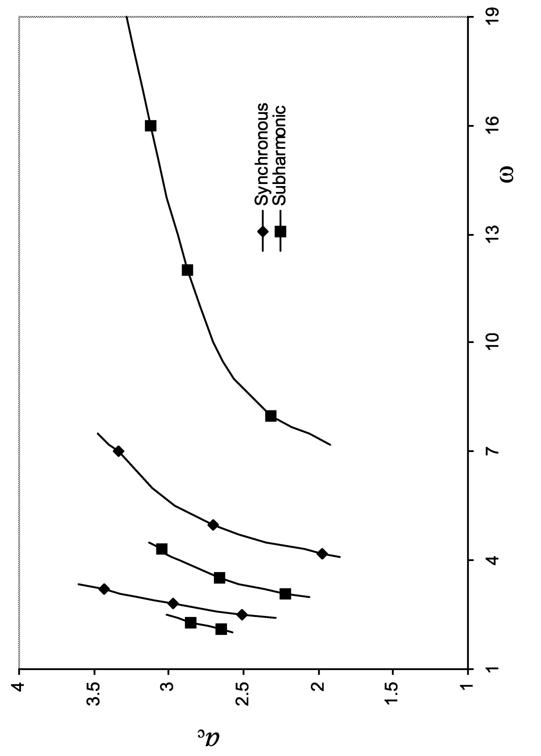


Fig. 4. Variation of $a_c$ with $\omega$. $\varepsilon = 0.1$, $P = 0.1$.

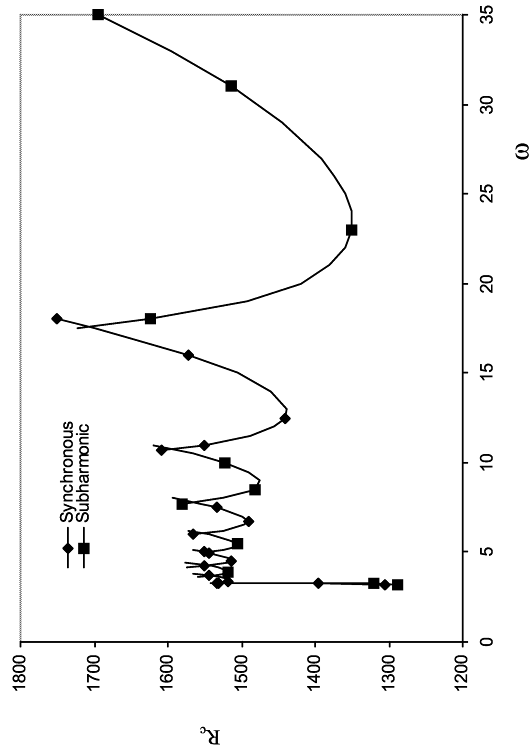


Fig. 5. Variation of $R_c$ with $\omega$. $\varepsilon = 0.1$, $P = 0.73$.

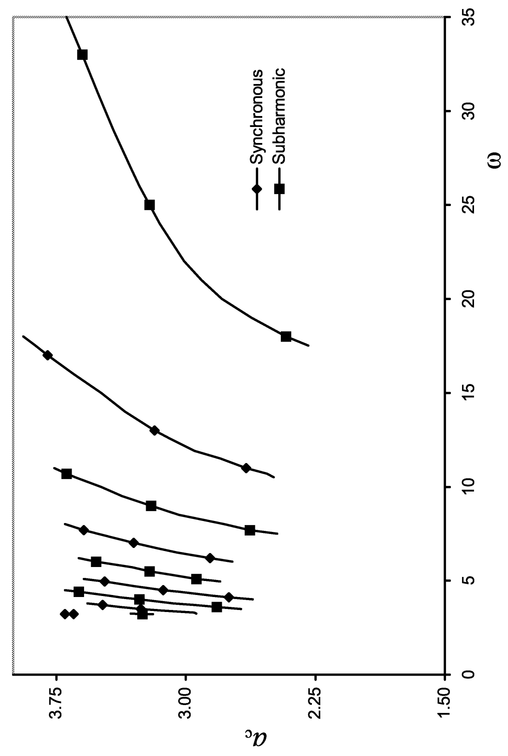


Fig. 6. Variation of $a_c$ with $\omega$. $\varepsilon = 0.1$, $P = 0.73$.

where $a_k$ and $b_k$ are as given above. Now, using the temperature profile (2.7), we write the externally imposed wall temperatures; when the temperature of the lower boundary as well as of the upper boundary is modulated:

$$T(t) = \beta d \left[ 1 + \varepsilon \left\{ \sum_{k=1}^{\infty} a_k \cos k\omega t + \sum_{k=1}^{\infty} b_k \sin k\omega t \right\} \right]$$

$$\text{at } z = -d/2. \tag{2.8a}$$

$$= \beta d \varepsilon \left\{ \sum_{k=1}^{\infty} a_k \cos k\omega t + \sum_{k=1}^{\infty} b_k \sin k\omega t \right\}$$

$$\text{at } z = d/2. \tag{2.8b}$$

Here $\varepsilon$ represents the amplitude of modulation and $\beta(= \Delta T/d)$ is the thermal gradient. The equations (2.1)–(2.4) and (2.8) admit an equilibrium solution in which

$$\mathbf{V} = (u, v, w) = 0, \quad T = \bar{T}(z,t), \quad p = \bar{p}(z,t). \tag{2.9}$$

The equation for the pressure $\bar{p}(z,t)$, which balances the buoyancy force, is not required explicitly, however the temperature $\bar{T}(z,t)$ can be given by the diffusion equation

$$\frac{\partial \bar{T}}{\partial t} = \kappa \frac{\partial^2 \bar{T}}{\partial z^2}. \tag{2.10}$$

The differential equation (2.10) can be solved with the help of the boundary conditions (2.8). Consider

$$\bar{T}(z,t) = T_S(z) + \in T_1(z,t), \tag{2.11}$$

where $T_S(z)$ is the steady temperature field and $T_1(z,t)$ is the oscillating part. Then the solution of (2.10) is

$$T_S(z) = \Delta T \left( \frac{1}{2} - \frac{z}{d} \right) \tag{2.12}$$

and

$$T_1(z,t) = -\Delta T \left[ \mathrm{Re} \left\{ \sum_{k=1}^{\infty} a_k F_k(z,t) \right\} + \mathrm{Im} \left\{ \sum_{k=1}^{\infty} b_k F_k(z,t) \right\} \right], \tag{2.13}$$

where

$$F_k(z,t) = \frac{\sinh(\lambda_k z/d)}{\sinh(\lambda_k/2)} e^{ik\omega t} \tag{2.14}$$

and

$$\lambda_k^2 = ik\omega d^2/\kappa. \tag{2.15}$$

Here the object is to examine the behaviour of infinitesimal disturbances to the basic solution (2.9). With this in view, substitute

$$\mathbf{V}(u,v,w,T) = \bar{T}(z,t) + \theta, p = \bar{p}(z,t) + p_1 \tag{2.16}$$

into (2.1)–(2.3) and linearize with respect to the perturbation quantities $\mathbf{V}$, $\theta$ and $p_1$. These quantities are Fourier analyzed with respect to their variations in $XY$-plane; we write

$$w = w(z,t) \exp[i(a_x x + a_y y)], \tag{2.17}$$

$$\theta = \theta(z,t) \exp[i(a_x x + a_y y)]. \tag{2.18}$$

Here $a = (a_x^2 + a_y^2)^{1/2}$ is the horizontal wave number. The variables have been non-dimensionalized according to

$$\begin{aligned} &\mathbf{r} = d\mathbf{r}', t = \tau/\omega, \bar{T} = \beta d T_0, \\ &\theta = \beta d \theta', a_x^2 + a_y^2 = d^2 a'^2 \\ &\mathbf{V} = (\alpha g \beta d^3 a^2/\nu)\mathbf{V}', \\ &p_1 = (\alpha g \beta \kappa d^2 \rho_m/\nu) p', \end{aligned} \tag{2.19}$$

where $\mathbf{r} = (x,y,z)$. Then the non-dimensionalized linear governing equations are

$$a^2 \omega^* \frac{\partial \mathbf{V}}{\partial \tau} + \nabla p = P\theta \hat{k} + a^2 P \nabla^2 \mathbf{V}, \tag{2.20}$$

$$\nabla \mathbf{V} = 0, \tag{2.21}$$

$$a^2 \omega^* \frac{\partial \theta}{\partial \tau} + Ra^2 \left( \frac{\partial T_0}{\partial z} \right) w = \nabla^2 \theta, \tag{2.22}$$

where $P = \nu/\kappa$ is the Prandtl number, $R = \alpha g \Delta T d^3/\nu\kappa$ is the Rayleigh number, $\hat{k}$ the vertical unit vector in the positive $z$ direction, and $\omega^* = \omega d^2/\kappa$ the non-dimensional frequency, which is a measure of the thickness of the thermal boundary layer at the planes. In the above equations the primes have been omitted.

The temperature gradient $\frac{\partial T_0}{\partial z}$, obtained from the dimensionless form of (2.11) is

$$\frac{\partial T_0}{\partial z} = -1 - \varepsilon \left[ \mathrm{Re} \left\{ \sum_{k=1}^{\infty} a_k F_k'(z,\tau) \right\} + \mathrm{Im} \left\{ \sum_{k=1}^{\infty} b_k F_k'(z,\tau) \right\} \right], \tag{2.23}$$

where

$$F_k'(z,\tau) = \frac{\lambda_k \cosh(\lambda_k z)}{\sinh(\lambda_k/2)} e^{ik\tau}, \tag{2.24}$$

and

$$\lambda_k^2 = ik\omega^*. \tag{2.25}$$

Henceforth the asterisk will be dropped and $\omega$ will be considered as the non- dimensional frequency. For convenience, the entire problem has been expressed in terms of $w$ and $\theta$. Taking the curl of (2.20) twice and using (2.17) and (2.18), the system of equations reduces to

$$\omega\left(\frac{\partial^2}{\partial z^2} - a^2\right)\frac{\partial w}{\partial \tau} = -P\theta + P\left(\frac{\partial^2}{\partial z^2} - a^2\right)^2 w, \tag{2.26}$$

$$\omega\frac{\partial \theta}{\partial \tau} = \left(\frac{\partial^2}{\partial z^2} - a^2\right)\theta - Ra^2\left(\frac{\partial T_0}{\partial z}\right) w. \tag{2.27}$$

The boundary conditions on $w$ and $\theta$ are

$$w = \frac{\partial w}{\partial z} = 0 \quad \text{at} \quad z = \pm\frac{1}{2}, \tag{2.28}$$

$$\theta = 0 \quad \text{at} \quad z = \pm\frac{1}{2}. \tag{2.29}$$

## 3. Method

From the expression (2.14) it is clear that $F_k(z,\tau)$ is an odd function of $z$, so $F_k'(z,\tau)$ is an even function of $z$. By carefully analyzing the equations (2.26) and (2.27) and the boundary conditions (2.28) and (2.29) one can see that the proper solution of the equations (2.26) and (2.27) can be divided into two non-combining groups of even and odd solutions. Previous investigations [20, 21] on thermal convection have shown that disturbances corresponding to even solutions are most unstable, therefore here the stability of the disturbances corresponding to the even eigenfunctions have been considered.

Now, since $\theta$ vanishes at $z = \pm\frac{1}{2}$, it is expanded in a series of $\cos[(2n+1)\pi z]$. Also $w$ is written in a series of $\phi_n$ so that

$$\left(\frac{\partial^2}{\partial z^2} - a^2\right)^2 \phi_n = \cos\left[(2n+1)\pi z\right], \tag{3.1}$$

where

$$\phi_n = \frac{\partial \phi_n}{\partial z} = 0 \quad \text{at} \quad z = \pm\frac{1}{2}. \tag{3.2}$$

The general solution of (3.1) is given by Chandrasekhar [1], p. 56:

$$\phi_n = P_n \cosh az + Q_n z \sinh az + \gamma_n^2 \cos[(2n+1)\pi z], \tag{3.3}$$

where

$$P_n = -(-1)^n \frac{(2n+1)\pi\gamma_n^2}{a+\sinh a}\sinh(a/2), \tag{3.4}$$

$$Q_n = (-1)^n \frac{2(2n+1)\pi\gamma_n^2}{a+\sinh a}\cosh(a/2), \tag{3.5}$$

and

$$\gamma_n = \frac{1}{(2n+1)^2\pi^2 + a^2}. \tag{3.6}$$

The expansions of $w$ and $\theta$ can be written as

$$w(z,\tau) = \sum_{n=0}^{\infty} A_n(\tau)\phi_n(z), \tag{3.7}$$

$$\theta(z,\tau) = \sum_{n=0}^{\infty} B_n(\tau)\cos[(2n+1)\pi z]. \tag{3.8}$$

Now substitute (3.7) and (3.8) into (2.26) and (2.27), and multiply by $\cos[(2m+1)\pi]$. The resulting equations are then integrated with respect to $z$ in the interval $\left(-\frac{1}{2},\frac{1}{2}\right)$. The outcome is a system of ordinary differential equations for the unknown coefficients $A_n(\tau)$ and $B_n(\tau)$

$$\begin{aligned}\omega\sum_{n=0}^{\infty}\left[K_{nm} - a^2P_{nm}\right]\frac{dA_n}{d\tau} &= \\ -\frac{P}{2}B_{\mathrm{m}} + P\sum_{n=0}^{\infty}&\left[L_{nm} - 2a^2K_{nm} + a^4P_{nm}\right]A_n,\end{aligned} \tag{3.9}$$

$$\begin{aligned}\frac{\omega}{2}\frac{\mathrm{d}B_{\mathrm{m}}}{\mathrm{d}\tau} &= -\frac{1}{2}[(2m+1)^2\pi^2 + a^2]B_{\mathrm{m}} + Ra^2\sum_{n=0}^{\infty}\Big[P_{nm} \\ &+\varepsilon\Big\{\mathrm{Re}\sum_{k=1}^{\infty}a_kH_{nmk}(\tau) + \mathrm{Im}\sum_{k=1}^{\infty}b_kH_{nmk}(\tau)\Big\}\Big]A_n \\ &(m = 0,1,2,\ldots).\end{aligned} \tag{3.10}$$

The other coefficients which occur in (3.9) and (3.10) are

$$H_{nmk}(\tau) = \frac{\lambda_k}{\sinh(\lambda_k/2)}G_{nmk}e^{ik\tau}, \tag{3.11}$$

$$P_{nm}=\int_{-1/2}^{1/2}\phi_n(z)\cos[(2m+1)\pi z]\mathrm{d}z, \qquad (3.12)$$

$$K_{nm}=\int_{-1/2}^{1/2}D^2\phi_n(z)\cos[(2m+1)\pi z]\mathrm{d}z, \qquad (3.13)$$

$$L_{nm}=\int_{-1/2}^{1/2}D^4\phi_n(z)\cos[(2m+1)\pi z]\mathrm{d}z, \qquad (3.14)$$

where

$$G_{nmk}=\int_{-1/2}^{1/2}\phi_n(z)\cosh(\lambda_k z)\cos[(2m+1)\pi z]\mathrm{d}z. \qquad (3.15)$$

Here the values of the integrals (3.12)–(3.14) have been obtained in their closed forms, however (3.11) or (3.15) has been calculated numerically, using Simpson's (1/3)rd rule [17], p. 125. Thus

$$P_{nm}=\frac{1}{2}\gamma_n^2\delta_{nm}+(-1)^m(2m+1)\pi\gamma_m[2P_n\cosh(a/2)+Q_n\{\sinh(a/2)-4a\gamma_m\cosh(a/2)\}], \qquad (3.16)$$

$$K_{nm}=-\frac{1}{2}\gamma_n^2(2n+1)^2\pi^2\delta_{nm}+(-1)^m(2m+1)\pi\gamma_m[2(a^2P_n+2aQ_n)\cosh(a/2) + a^2Q_n\{\sinh(a/2)-4a\gamma_m\cosh(a/2)\}], \qquad (3.17)$$

$$L_{nm}=\frac{1}{2}\gamma_n^2(2n+1)^4\pi^4\delta_{nm}+(-1)^m(2m+1)\pi\gamma_m[2a^4P_n\cosh(a/2) + Q_n\{4a^3(2-a^2\gamma_m)\cosh(a/2)+a^4\sinh(a/2)\}], \qquad (3.18)$$

where $\delta_{nm}$ is the Kronecker delta.

It is convenient for computations to take $m=0,1,2,\ldots,N-1$, i.e. 2N equations and then rearrange them. For this, first multiply the Eq. (3.9) by the inverse of the matrix ($\mathrm{K}_{nm}-a^2P_{nm}$), and then introduce the notations

$$x_1=A_0,\ x_2=B_0,\ x_3=A_1,\ x_4=B_1,\ \ldots. \qquad (3.19)$$

Now combine the equations (3.9) and (3.10) to the form

$$\frac{dx_i}{d\tau}=H_{i1}x_1+H_{i2}x_2+\cdots+H_{iL}x_L \quad (i=1,2,3,\ldots,2N \text{ and } L=2N), \qquad (3.20)$$

where $H_{ij}(\tau)$ is the matrix of the coefficients in the Eqs. (3.9) and (3.10).

## 4. Analysis

Since the coefficients $H_{ij}(\tau)$ of the equations (3.20) are either constant or periodic in $\tau$ with the period $\tau_0=2\pi/\omega$, the stability of the solution of (3.20) can be discussed on the basis of the Floquet theory [16], p. 55. The solution in the Floquet theory must be of the form $e^{\mu\tau}P(\tau)$, where $P(\tau)$ is a periodic function of $\tau$ with the period $\tau_0$. The vanishing of the real part $\mu_\mathrm{r}$ of $\mu$ gives the stability boundary. Here our aim is to determine the condition under which $\mu_\mathrm{r}=0$.

The disturbance is synchronous with the unsteady part of the mean temperature field, if the imaginary part $\mu_\mathrm{i}$ of $\mu$ is zero. However if $\mu_\mathrm{i}\tau_0$ is equal to $\pi$ or $-\pi$, then the disturbance is subharmonic, having a frequency half that of the unsteady mean temperature field. Let

$$x_n(\tau)=x_{in}(\tau)=\mathrm{col}[x_{1n}(\tau),x_{2n}(\tau),\ldots,x_{Ln}(\tau)] \quad (n=1,2,3,\ldots,2N) \qquad (4.1)$$

be the solutions of (3.20) which satisfy the initial conditions

$$x_{in}(0)=\delta_{in}. \qquad (4.2)$$

The solutions (4.1) with the conditions (4.2) form $2N$ linearly independent solutions of the equation (3.20). Once these solutions are found, one can get the values of $x_{in}(2\pi)$ and then arrange them in the constant matrix

$$C=[x_{in}(2\pi)]. \qquad (4.3)$$

The eigenvalues $\lambda_1,\lambda_2,\lambda_3,\ldots,\lambda_L$ of the matrix $C$ are also called the characteristic multipliers of the system (3.20), and the numbers $\mu_\mathrm{r}$, defined by the relations

$$\lambda_j=\exp(2\pi\mu_j), \quad j=1,2,3,\ldots,2N \qquad (4.4)$$

are the characteristic exponents.

The values of the characteristic exponents determine the stability of the system. We assume that the $\mu_j$ are ordered so that

$$\mathrm{Re}(\mu_1) \geq \mathrm{Re}(\mu_2) \geq \cdots \geq \mathrm{Re}(\mu_L). \quad (4.5)$$

Then the system is stable if $\mathrm{Re}(\mu_1) < 0$, while $\mathrm{Re}(\mu_1) = 0$ corresponds to one periodic solution and represents a stability boundary. This periodic disturbance is the only disturbance, which will manifest itself as marginal stability.

The fourth order Runge-Kutta-Gill method [17], p. 217, has been used to integrate the system (3.20), and the matrix $C$ is obtained. We have used Rutishauser method [18], p. 116, to find the eigenvalues $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_L$ of the matrix $C$.

## 5. Results and Discussion

The first approximation of the Rayleigh number for the onset of convection, in the absence of modulation ($\in = 0$), is found by setting $n = 0$, $m = 0$ in (3.9) and (3.10). This corresponds to $\cos \pi z$, a trial function for $\theta$. The corresponding value of $R$ is

$$R = \frac{(\pi^2 + a^2)^3}{a^2 \left[1 - 16 a \pi^2 \cosh^2(a/2) / \{(\pi^2 + a^2)^2 (\sinh a + a)\}\right]} \quad (5.1)$$

which gives the minimum value as

$$R_{\mathrm{neut}} = 1715.08 \text{ for } a = 3.114. \quad (5.2)$$

This is in contrast to the exact value 1707.76, at the wavenumber 3.117. The second approximation to the Rayleigh number is found to be 1707.9375 at $a = 3.11621$, which is obtained by setting $m$, $n = 0$ and 1. Similarly the third approximation, obtained by putting $n$, $m = 0, 1, 2$ is 1707.776 at $a = 3.11631$. These values are equal, as they should, to Chandrasekhar's [1] values. By including more terms in the expansion of $w$ and $\theta$ one can achieve a higher degree of accuracy. With the help of the available packages one could carry out these calculations more accurately.

When $\in \neq 0$, we calculate the modified value of the critical Rayleigh number $R_c$, with variation in other parameters. We also check the critical value of the wavenumber $a$. Here the results have been obtained by solving the equations (3.20) for $x_1$, $x_2$, $x_3$ and $x_4$. The results are calculated for moderate values of $\in$, as we are interested only in the modulating effect of the oscillation, there seems to be no reason why this theory can not be applied for large values of the parameters.

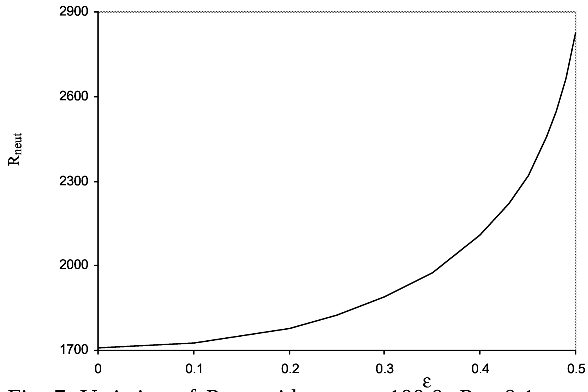


Fig. 7. Variation of $R_{\mathrm{neut}}$ with $\varepsilon$. $\omega = 100.0$, $P = 0.1$, $a = 3.116846$.

Here it is possible to obtain a relationship between the critical Rayleigh number $R_c$ and the corresponding critical wavenumber $a_c$ in terms of the other parameters $P$, $\in$ and the dimensionless frequency $\omega$. The critical Rayleigh number is the minimum value of $R$ as a function of the wavenumber $a$ with variation in the parameters $P$, $\in$ and $\omega$. The corresponding value of $a$ is known as the critical wavenumber $a_c$. To obtain the critical curve $R_c$ versus $\omega$ (Figs. 3, 5) we proceed as follows: First fix $P$ and $\in$, then for some value of $\omega$, find that value of the wavenumber for which $R$ is minimum. This minimum value of $R$ and the corresponding value of the wavenumber are the critical Rayleigh number and the critical wavenumber, respectively. Continuing in this way we find $R_c$ and the corresponding $a_c$ for different values of $\omega$. Thus we get two curves $R_c$ versus $\omega$ and $a_c$ versus $\omega$, for fixed values of $P$ and $\in$. Each of these two curves consists of different curves, corresponding to synchronous solutions(S-curves) having the same frequency as the applied temperature field and corresponding to a subharmonic one(H-curve) having half the frequency of the applied temperature field.

From Figs. 3, 5 one can see that each cusp in the $R_c$-$\omega$ curve is really the intersection of an S-curve and an H-curve, both of which can be continued beyond their intersections. Thus in the area above the H-curve there are also synchronous disturbances, but disturbances with half-frequency can be expected to be more unstable. Similarly above an S-curve there are disturbances with half frequency but synchronous disturbances are more unstable. The critical wavenum-

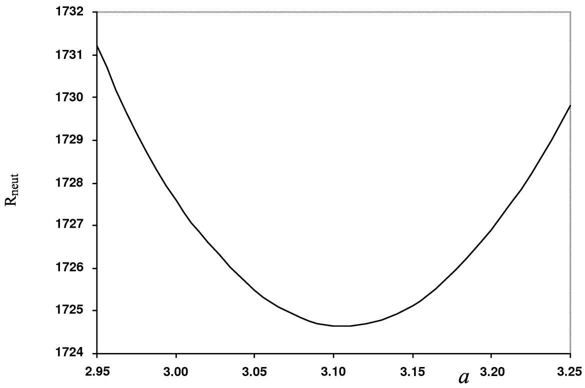


Fig. 8. Variation of $R_{\text{neut}}$ with $a$. $\omega = 100.0$, $P = 0.1$, $\varepsilon = 0.1$.

ber appears to be discontinuous from the $a_c$-$\omega$ curves (Figs. 4, 6) only because S-curves and H-curves are not continued beyond their intersections. The $a_c$-$\omega$ curves are also composed of S-curves and H-curves. The existence of synchronous and subharmonic disturbances has already been indicated by Gresho and Sani [8], Yih and Li [7], Clever et al. [19] and Aniss et al. [11] in their investigations.

From Fig. 3 it is clear that the effect of modulation on the thermal instability is one of stabilization as convection occurs at higher Rayleigh number than in the unmodulated case ($\varepsilon = 0$, $R_c = 1707.9375$). This stabilizing effect decreases with increasing frequency. Also Fig. 5 shows that the modulation destabilizes the system; with convection occurring at an earlier point than in the steady case. This is because convective waves propagate across the fluid layer, thereby inhibiting the instability, and so convection occurs at a higher Rayleigh number (Fig. 3) than that predicted by the theory with steady temperature gradient (5.2). However, when the Prandtl number increases, the viscous term dominates and so convection occurs at an early point (Fig. 5).

The above results agree with the results of Venezian [2] and Bhatia and Bhadauria [15]. They found that, when the temperature modulation is out of phase and the Prandtl number is small, the effect of modulation is stabilizing. However at high Prandtl number this effect is destabilizing. The results also agree with those of Bhadauria [18], and Yih and Li [7] who found, while studying the instability of unsteady flow, that the effect of modulation is destabilizing. They also found that the critical curve is composed of two curves, one corresponding to the synchronous solution and the other corresponds to the subharmonic one.

As predicted by Yih and Li [7], here it is found that when the modulating frequency is small there is a sudden fall in the value of the critical Rayleigh number (Figs. 3, 5).

The value of $R$ at neutral stability ($R_{\text{neut}}$), with respect to the amplitude of a modulation $\varepsilon$, is shown in Fig. 7. It is found that, as the amplitude of the modulation increases, $R_{\text{neut}}$ also increases, showing the stabilizing effect. This agrees with the findings of Rosenblat and Tanaka [5], where their $R_c$ increases monotonically with $\varepsilon$.

In Fig. 8 we depict the variation of $R_{\text{neut}}$ with the wavenumber $a$. From the figure it is clear that the critical value of the wavenumber $a$ is very close to 3.11.

*Acknowledgement*

This work was carried out as part of the work of Major Research Project awarded by University Grants Commission(UGC), India. The financial assistance by UGC is gratefully acknowledged. Part of the work was completed during my tenure as a faculty of Mathematics, Eritrean Institute of Technology, Asmara (N. E. Africa).

[1] S. Chandrasekhar, Hydrodynamic and Hydromagnetic Stability, Oxford University Press, London 1961.
[2] G. Venezian, J. Fluid Mech. **35**, 243 (1969).
[3] R. J. Donnelly, Proc. Roy. Soc. A **281**, 130 (1964).
[4] S. Rosenblat and D. M. Herbert, J. Fluid Mech. **43**, 385 (1970).
[5] S. Rosenblat and G. A. Tanaka, Phys. Fluids **14**, 1319 (1971).
[6] G. Z. Gershuni and E. M. Zhukhovitskii, J. Appl. Math. Mech. **27**, 1197 (1963).
[7] C.-S. Yih and C.-H. Li, J. Fluid Mech. **54**, 143 (1972).
[8] P. M. Gresho and R. L. Sani, J. Fluid Mech. **40**, 783 (1970).
[9] R. G. Finucane and R. E. Kelly, Int. J. Heat Mass Transfer **19**, 71 (1976).
[10] M. N. Roppo, S. H. Davis, and S. Rosenblat, Phys. Fluids **27**, 796 (1984).
[11] S. Aniss, M. Souhar, and M. Belhaq, Phys. Fluids **12**, 262 (2000).
[12] B. S. Bhadauria, Z. Naturforsch. **57a**, 780 (2002); Z. Naturforsch. **58a**, 176 (2003).
[13] B. S. Bhadauria and P. K. Bhatia, Physica Scripta **66**, 59 (2002).

[14] P. K Bhatia and B. S. Bhadauria, Z. Naturforsch. **56a**, 509 (2001).

[15] P. K. Bhatia and B. S. Bhadauria, Z. Naturforsch. **55a**, 957 (2000).

[16] L. Cesari, Asymptotic Behavior and Stability problems, Springer Verlag, Berlin 1963.

[17] S. S. Sastry, Introductory Methods of Numerical Analysis, 2nd Ed. Prentice-Hall of India Private Limited, New Delhi 1993.

[18] M. K. Jain, S. R. K. Iyengar, and R. K. Jain, Numerical Methods for Scientific and Engineering Computation, Wiley Eastern Limited, New Delhi 1991.

[19] R. Clever, G. Schubert, and F. H. Busse, J. Fluid Mech. **253**, 663 (1993).

[20] A. Pellew and R. V. Southwell, Proc. Roy. Soc. (London) **A 176**, 312 (1940).

[21] W. H. Reid and D. L. Harris, Phys. Fluids **1**, 102 (1958).